

Analog Computer Applications
Application Note #7
Undergraduate Engineering Physics Space-Themed STEM Project

Navigating the Slopes of a Methane Frost-Capped Mountain on Pluto
(A Simulation)

1 Introduction

This application note simulates the speed of a drone as it travels down a mountain slope covered with methane frost. Initially, the drone is at rest atop the slope. Once its brake is released, the drone begins to descend under the influence of atmospheric drag (assumed to be quadratic) and sliding friction. In section 2, a mathematical model is developed that describes the drone's speed as a function of time.

For comparison, analytical, modified Euler method, and analog computation solutions will be determined and displayed in table 1.

2 Mathematical modeling

The following mathematics will be a bit lengthy, so you may want to grab a snack, some iced tea (stirred, not shaken), a comfortable chair, and hang on tight!

Starting with Newton's Second Law of Motion,

$$ma = F_{\text{net}} = -\text{drag} - \text{friction} + \text{slope component of force due to gravity}$$

$$ma = -kv^2 - \mu mg \cos(\theta) + mg \sin(\theta)$$

$$a = -k/m v^2 + [\sin(\theta) - \mu \cos(\theta)]g$$

Letting $a = dv/dt$ and $g' = [\sin(\theta) - \mu \cos(\theta)]g$,

$$dv/dt = -k/m v^2 + g' \tag{1}$$

$$dv/dt = -k/m(v^2 - mg'/k)$$

Letting $dv/dt = 0$ to determine the drone's terminal speed,

$$0 = -k/m(v_T^2 - mg'/k)$$

$$v_T = \sqrt{(mg'/k)} \quad (2)$$

Returning to (1), which is separable and integrable,

$$dv/(v^2 - mg'/k) = -k/m dt$$

Using (2),

$$k/m = g'/v_T^2$$

$$dv/(v^2 - v_T^2) = -k/m dt \text{ with } v(0) = 0$$

$$\int_0^v dv/(v^2 - v_T^2) = -g'/v_T^2 \int_0^t dt \quad (3)$$

Using the method of partial fractions for the integrand on the left side of (3),

$$1/(v^2 - v_T^2) = A/(v - v_T) + B/(v + v_T) \quad (4)$$

After a bit of algebra,

$$0v + 1 = (A + B)v + (A - B)v_T$$

Comparing coefficients of like terms,

$$(A + B) = 0 \text{ or } B = -A$$

and

$$(A - B)v_T = 1 = (A + A)v_T = 1 \text{ or } A = 1/(2v_T)$$

Substituting the expressions for A and B into (4),

$$1/(v^2 - v_T^2) = 1/(2v_T)[1/(v - v_T) - 1/(v + v_T)] \quad (5)$$

Substituting (5) into the integrand on the left side of (3),

$$1/(2v_T) \int_0^v \left[\frac{1}{(v - v_T)} - \frac{1}{(v + v_T)} \right] dv = -g'/v_T^2 \int_0^t dt \quad (6)$$

Integrating (6) by inspection, and noting that $v < v_T$,

$$1/(2v_T)[\ln((v - v_T)/-v_T) - \ln((v + v_T)/v_T)] = -g't/v_T^2$$

Using properties of logarithms,

$$\ln[-(v - v_T)/(v + v_T)] = -2g't/v_T$$

$$-(v - v_T)/(v + v_T) = e^{(-2g't/v_T)}$$

After some algebra,

$$v = v_T[1 - e^{(-2g't/v_T)}]/[1 + e^{(-2g't/v_T)}]$$

Multiplying the numerator and denominator by $e^{(g't/v_T)}$ (a cute 'trick'),

$$v = v_T[e^{(g't/v_T)} - e^{(-g't/v_T)}]/[e^{(g't/v_T)} + e^{(-g't/v_T)}]$$

Noting that

$$2\sinh(g't/v_T) = e^{(g't/v_T)} - e^{(-g't/v_T)}$$

and

$$2\cosh(g't/v_T) = e^{(g't/v_T)} + e^{(-g't/v_T)},$$

then

$$v = v_T \sinh(g't/v_T) / \cosh(g't/v_T)$$

$$v = v_T \tanh(g't/v_T) \tag{7}$$

Assuming $g = 0.62 \text{ m/s}^2$, $\theta = 23^\circ$, $m = 4.00 \text{ kg}$, $k = 0.08 \text{ kg/m}$, and $\mu = 0.10$,

$$g' = [\sin(23^\circ) - 0.10\cos(23^\circ)]0.62 \text{ m/s}^2 = 0.185 \text{ m/s}^2$$

and

$$v_T = \sqrt{(4 \text{ kg} \times 0.185 \text{ m/s}^2 / 0.08 \text{ kg/m})} = 3.04 \text{ m/s}$$

Rewriting (1), using the above parameters, and omitting units for clarity,

$$dv/dt = -0.020v^2 + 0.185 \tag{8}$$

Well, you were warned that the math would be a bit lengthy! 😊

3 Computer setup

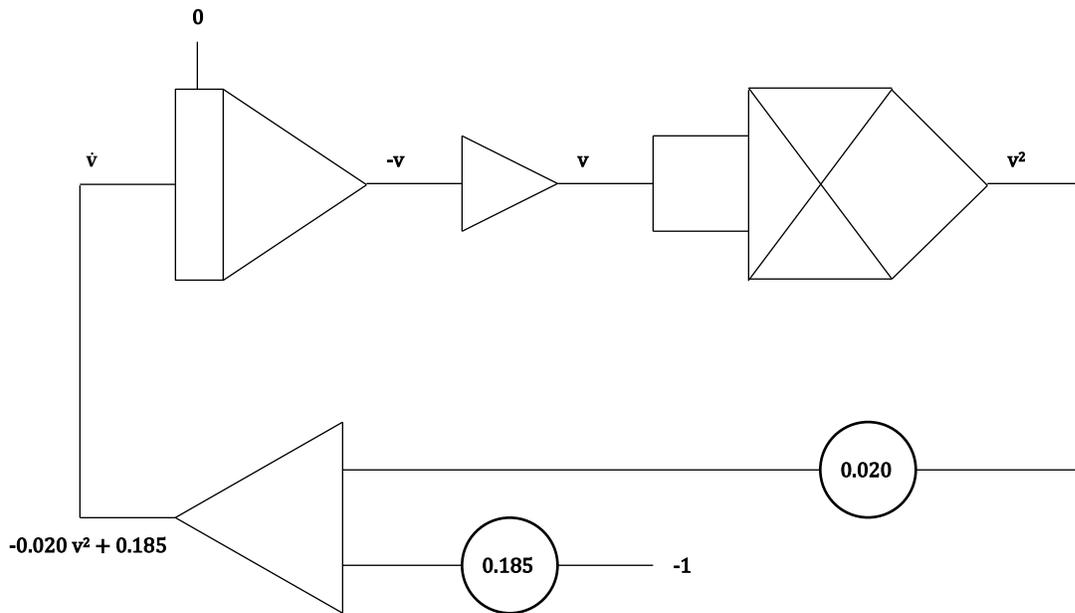


Figure 1: Computer setup for navigating the slopes

Modified Euler method
(hand-held programmable calculator)

```

PROGRAM:PLUTOV
:ClrHome:ClrDraw
:"DV/DT=-0.020V2 +0.185"
:"INITIAL CONDITION:"
:0→T:0→V
:"STEP SIZE:"1→H
:Fix 2
:Lbl 1
:If T>25:Then
:Goto 2:Else
:Disp {T,V}
:-0.020V2+0.185→F
:T+H→T
:V+FH→W
:-0.020W2+0.185→S
:(F+S)/2→A
:V+AH→V
:Pause:Goto1
:Lbl 2:End

```

4 Results

| Time t(s) | Analog Computation Solution estimated from oscilloscope v(m/s) | Analytical Solution v(m/s) | Numerical Method Solution from hand-held calculator v(m/s) |
|--------------|--|-------------------------------|--|
| 00.0 | 0.0 | 0.00 | 0.00 |
| 10.0 | 1.6 | 1.65 | 1.65 |
| 20.0 | 2.5 | 2.55 | 2.55 |
| 30.0 | 2.8 | 2.89 | 2.89 |
| 40.0 | 2.9 | 2.99 | 2.99 |
| 50.0 | 3.0 | 3.03 | 3.03 |
| >50 | terminal speed = 3.0 | 3.04 | 3.04 |

Table 1: Solution Comparisons

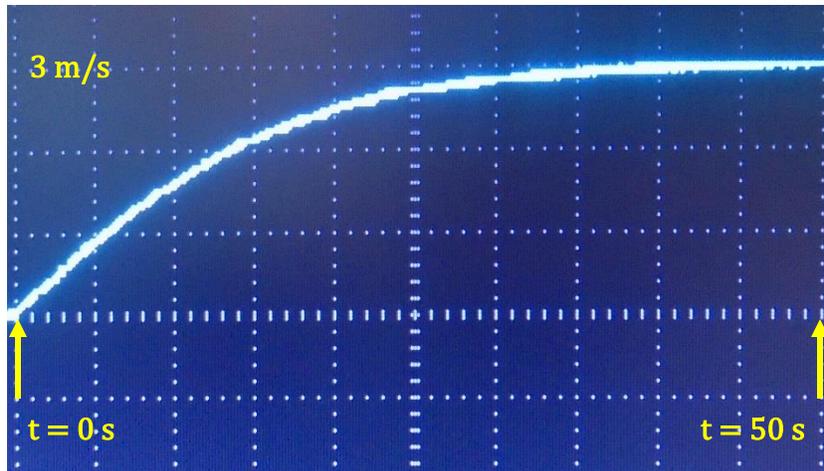


Figure 2: Drone speed as a function of time*



Figure 3: Drone terminal speed ($t > 50$ s, not quite infinity!) via digital multimeter

*For this application note, the display was produced during a single run by a differential equation analog computer prototype using discrete components with tolerances between 1% and 10%.